A stochastic view of Caffarelli-Silvestre theorem

Michelangelo Cavina

4 march 2022

References:

- B. Øksendal. Stochastic differential equations
- S. A. Molchanov, E. Ostrovskii. Symmetric stable processes as traces of degenerate diffusion processes
- A. Göing-Jaeschke, M. Yor. A survey and some generalizations of Bessel processes
- L. Caffarelli, L. Silvestre. An extension problem related to the fractional Laplacian

In this section we give an introduction on some notions from the theory of stochastic analysis, which are useful to solve Dirichlet problems, and some examples of their applications.

References [Øksendal, chapters 7 and 9]

$\boxed{\text{Notations}}$ We will denote by $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ a filtered probability space Ω of σ-algebra $\mathcal{F}$, probability measure P and filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We will omit writing the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ because we will always use the natural filtration associated to the Brownian motion

$\boxed{\text{Def}}$ (Itô diffusion)

A (time homogeneous) Itô diffusion is a stochastic process

$$X : [0,+\infty) \times \Omega \longrightarrow \mathbb{R}^n$$
$$(t,\omega) \longmapsto X_t(\omega)$$

satisfying the following stochastic differential equation

$$\underset{n\times 1}{dX_t} = \underset{n\times 1}{b(X_t)}\,\underset{1\times 1}{dt} + \underset{n\times m}{\sigma(X_t)}\,\underset{m\times 1}{dB_t} \;;\quad X_0 = x \qquad (E1)$$

where $x \in \mathbb{R}^n$ is the starting point at time $t=0$,
$B = \{B_t\}_{t \geq 0}$ is a standard m-dimensional Brownian motion, and

$$b : \mathbb{R}^n \longrightarrow \mathbb{R}^n \ ; \qquad \sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m}$$

are coefficients satisfying proper conditions [Øksendal, chapter 7]
We recall that (E1) can be written in the integral form

$$X_t = x + \int_0^t b(X_v)\,dv + \int_0^t \sigma(X_v)\,dB_v \qquad (E1')$$

$\boxed{\text{Notation}}$ We will denote a stochastic process X by $X = \{X_t\}_{t \geq 0}$. We will denote by $\{X_t^x\}_{t \geq 0}$ a Itô diffusion X of starting point $x \in \mathbb{R}^n$.

Given $X : [0,+\infty) \times \Omega \longrightarrow \mathbb{R}^n$ a stochastic process

- Let $t \in [0,+\infty)$. Then
$$X_t : \Omega \longrightarrow \mathbb{R}^n$$
$$\omega \longmapsto X_t(\omega) := X(t,\omega)$$

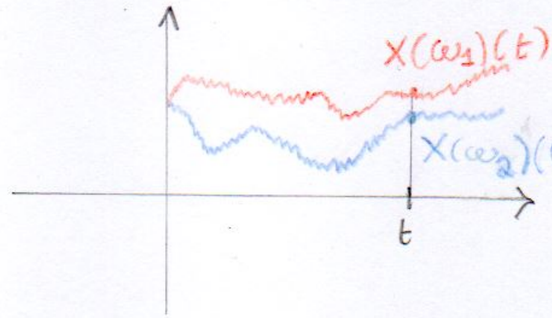
is a random variable. Moreover, if
$\tau : \Omega \longrightarrow [0,+\infty]$ is a stopping time, then

$$X_\tau : \Omega \longrightarrow \mathbb{R}^n \ ; \qquad X_\tau(\omega) := \begin{cases} X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < +\infty \\ 0 & \text{otherwise} \end{cases}$$

is a random variable.

- Let $\omega \in \Omega$. Then $$X(\omega) : [0,+\infty) \longrightarrow \mathbb{R}^n$$ $$t \longmapsto X(\omega)(t) := X(t,\omega)$$

is called a trajectory of X.



Def (infinitesimal generator)

Let $X = \{X_t\}_{t \geq 0}$ be an Itô diffusion in $\mathbb{R}^n$.

The infinitesimal generator A of X is defined by

$$Af(x) = \lim_{t \to 0^+} \frac{\mathbb{E}\left[f(X_t^x)\right] - f(x)}{t} \quad \text{for } x \in \mathbb{R}^n \quad (E2)$$

Remark This operator is well defined everywhere for functions $f \in C_0^2(\mathbb{R}^n)$.

Theorem (Characterization of infinitesimal generators)

Let $\{X_t\}_{t \geq 0}$ be a Itô diffusion satisfying

$$\underbrace{dX_t}_{n\times 1} = \underbrace{b(X_t)}_{n\times 1}\,\underbrace{dt}_{1\times 1} + \underbrace{\sigma(X_t)}_{n\times m}\,\underbrace{dB_t}_{m\times 1}$$

let $f \in C^2(\mathbb{R}^n)$. Then

$$Af(x) = \sum_{i=1}^{n} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \cdot \sigma^T)_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) \quad (E3)$$

Examples 1) Let $\{X_t\}_{t\geq 0}$ be a n-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$. Then

$dX_t = dB_t$. To be more precise: $\sigma(x) = I_n$, $b(x) = \vec{0}$ $\forall x \in \mathbb{R}^n$.

So $\sigma \cdot \sigma^T = I_n$. Hence the infinitesimal generator of $\{X_t\}_{t\geq 0}$ is

$$Af(x) = \sum_{i=1}^{n} 0 \cdot \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} \delta_{i,j} \cdot \frac{\partial^2}{\partial x_i \partial x_j} f(x) = \frac{1}{2} \Delta f(x)$$

So we proved that the infinitesimal generator of $\{B_t\}_{t\geq 0}$ is $\frac{1}{2}\Delta$.

2) Let $\{X_t\}_{t\geq 0}$ be the process in $\mathbb{R}^n \times \mathbb{R} \ni (x, v)$, solving

$$\underset{(n+1)\times 1}{\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ \vdots \\ dX_t^n \\ dX_t^{n+1} \end{pmatrix}} = \underset{(n+1)\times 1}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}} \underset{1\times 1}{dt} + \underset{(n+1)\times(n+1)}{\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right)} \cdot \underset{(n+1)\times 1}{\begin{pmatrix} dB_t^1 \\ dB_t^2 \\ \vdots \\ dB_t^n \\ dB_t^{n+1} \end{pmatrix}}$$

It follows that $\sigma \cdot \sigma^T = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array}\right)$, so the infinitesimal generator is

$$Af(x) = -\frac{\partial f}{\partial v}(x, v) + \frac{1}{2}\Delta_x f(x, v).$$

The infinitesimal generator is the heat operator.

Def (first exit time for a process)

Let $D \subseteq \mathbb{R}^n$ be a domain, let $X: [0,+\infty) \times \Omega \longrightarrow \mathbb{R}^n$ be a stochastic process. We denote by "first exit time of X from D" the random variable

$$\tau_D : \Omega \longrightarrow [0,+\infty] \quad ; \quad \tau_D(\omega) := \inf \{ t > 0 \mid X_t(\omega) \notin D \}$$

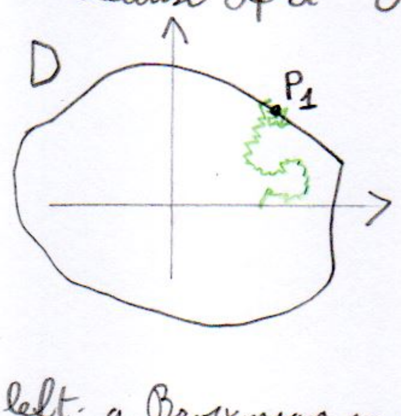
Def (regularity with respect to an Itô diffusion)

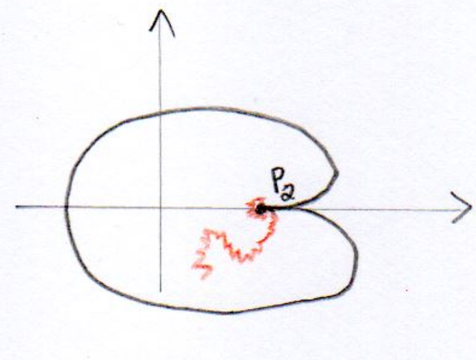
Under the previous notations, assume X is an Itô diffusion. We say that a point $y \in \partial D$ is regular w.r.t. X if

$$P^y[\tau_D = 0] = 1,$$

otherwise y is called irregular.

Remark $P^y[\tau_D = 0]$ is either equal to 1 or equal to 0, because of a 0-1 law.





On the left: a Brownian motion exits from D immediatly almost surely
P_1 is regular

On the right: the domain is a Lebesgue spine. The Brownian motion does not immediatly exit almost surely.
P_2 is irregular

$\boxed{\text{Theorem}}$ (stochastic solution of the Dirichlet problem)

Let $D \subseteq \mathbb{R}^n$ be a domain, let $u \in C(\partial D)$, u bounded.

Consider the Dirichlet problem

$$(\text{D.P.}) \quad \begin{cases} A w = 0 & \text{in } D \\ w(x) = u(x) & \text{for } x \in \partial D \end{cases}$$

Let $\{X_t\}_{t \geq 0}$ be an Itô diffusion such that the infinitesimal generator of $\{X_t\}_{t \geq 0}$ is A.

Consider the function

$$f : \overline{D} \longrightarrow \mathbb{R}^n$$

$$f(x) = \mathbb{E}^x\left[u(X_{\tau_D})\right] = \mathbb{E}\left[u(X^x_{\tau_D})\right] \quad (E4)$$

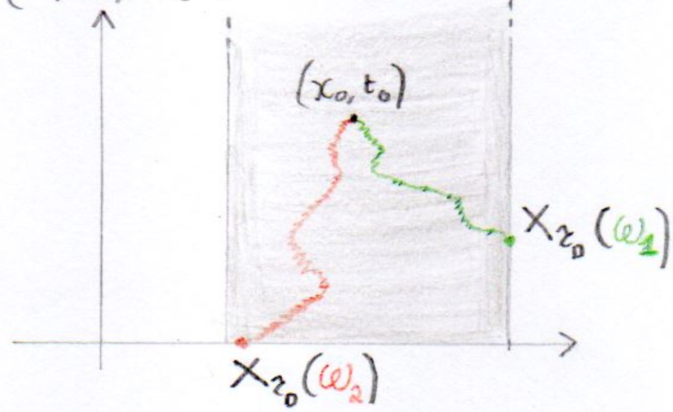
Then, under suitable hypotheses (A is uniformly elliptic in D, $\tau_D < +\infty$ almost surely, ...) [Øksendal, chapter 9],

f is a solution of

$$\text{D.P.}') \quad \begin{cases} Af = 0 & \text{in } D \\ f(x) = u(x) & \text{for } x \in \partial D,\ x \text{ regular w.r.t. } \{X_t\}_{t \geq 0}. \end{cases}$$

$\boxed{\text{Example}}$ (Heat equation in $\mathbb{R}^2$)

Consider the domain $(x,t) \in (a,b) \times (0,+\infty) = D$.



Let $\{X_t\}_{t \geq 0}$ be a process in $\mathbb{R}^2$ satisfying

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dB_t^1 \\ dB_t^2 \end{pmatrix}$$

This is the process considered in the example 2). We showed that the infinitesimal generator of $\{X_t\}_{t \geq 0}$ is the heat operator $A = \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$. Moreover, we can solve the equation to get

$$\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} B_t^1 \\ -t \end{pmatrix} + \begin{pmatrix} x_0 \\ t_0 \end{pmatrix}$$

The process behaves like a Brownian motion in the first variable and behaves like a constant drift of velocity -1 in the second variable.

Starting from $(x_0, t_0) \in D$ all the trajectories land on ∂D in a finite amount of time.

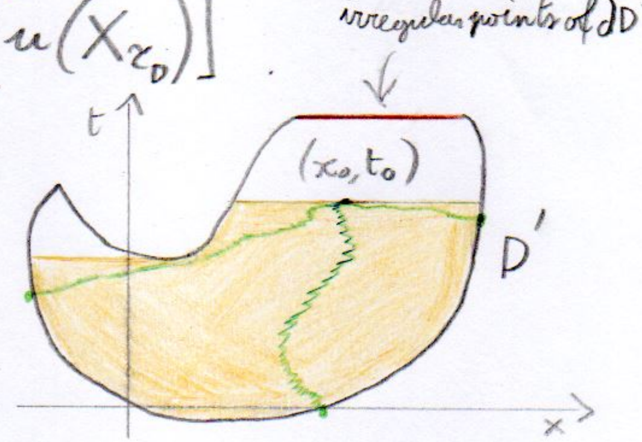
Moreover, all $y \in \partial D$ are regular w.r.t. $\{X_t\}_{t \geq 0}$.

So the hypotheses of theorem about the stochastic solution of the Dirichlet problem are satisfied.

to the Dirichlet problem

$$\begin{cases} \left(\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) w = 0 \quad \text{in } D \\ w(y) = u(y) \quad \text{on } \partial D \end{cases} \qquad \text{for } u \in C(D),\ u \text{ bounded}$$

has the solution $f(x,t) = \mathbb{E}^{(x,t)}\left[u(X_{\tau_D})\right]$.



Let us consider a domain $D' \subseteq \mathbb{R}^2$ like in the picture. We have a segment of points that are irregular w.r.t. $\{X_t\}_{t \geq 0}$ (the red points), so the solution $f(x,t) = \mathbb{E}^{(x,t)}\left[u(X_{\tau_D})\right]$ may not be equal to $u(y)$ when $(x,t) = y$ is in the set of irregular points in $\partial D'$.

Moreover, starting from (x_0, t_0), the trajectories of $\{X_t\}_{t \geq 0}$ can only move through the yellow region before hitting the boundary, so the solution $\mathbb{E}^{(x_0,t_0)}\left[u(X_{\tau_D})\right] = f(x_0, t_0)$ depends only on the yellow part of the boundary

(domain of dependence, H-trajectories)

The Bessel process

In this section we state some properties about the Bessel process, which will be used in the stochastic approach to Caffarelli - Silvestre theorem.

$\boxed{\text{Def}}$ (Bessel process) Let $0<s<1$. We denote by Bessel process in $\mathbb{R}$ the stochastic process

$$Z : [0,+\infty) \times \Omega_2 \longrightarrow \mathbb{R}$$
$$(t,\omega) \longmapsto Z_t(\omega)$$

satisfying the following stochastic differential equation

$$dZ_t = \frac{1-2s}{2Z_t}\,dt + dB_t^{n+1} \qquad (E6)$$

where B_t^{n+1} is a 1-dimensional Brownian motion

[Molchanov, Ostrovskii], [Göing-Jaeschke, Yor]

$\boxed{\text{Proposition}}$ The following facts about the Bessel process $\{Z_t\}_{t\geq 0}$ hold:

1) $\{Z_t\}_{t\geq 0}$ is a continuous diffusion process.

2) Let $z_0>0$ be a starting point. Then the trajectories $t\longmapsto Z_t^{z_0}(\omega)$ hit the point 0 in a finite amount of time almost surely

3) Let z_0 be a starting point. Then the random variable "first hitting time for the process $\{Z_t\}_{t\geq 0}$ going from z_0 to 0" admits a density with respect to the Lebesgue measure. The density function is

$$\Phi_{z_0}(t) = \chi_{(0,+\infty)}(t)\,\frac{1}{t\cdot\Gamma(s)}\left(\frac{z_0^2}{2t}\right)^s\cdot e^{-\frac{z_0^2}{2t}}$$

[Göing-Jaeschke, Yor. Page 8, equation (15)]

$\boxed{\text{Remark}}$ For any choice of $\alpha>0$ and $M>0$ we have

$$\int_0^{+\infty}\frac{1}{t\,\Gamma(\alpha)}\left(\frac{M}{2t}\right)^\alpha\cdot e^{-\frac{M}{2t}}\,dt = 1 \qquad (E8)$$

Proof: follows from the change of variables $\frac{M}{2t}\longleftrightarrow v$.

Caffarelli-Silvestre theorem and the Poisson Kernel

In this section we use the previous theorems about the stochastic Dirichlet problem to prove a formula, about a Poisson-type of Kernel, which entails the Caffarelli-Silvestre theorem.

$\boxed{\text{Def}}$ (fractional Laplacian)

Let $0<s<1$. Let $u \in C^2(\mathbb{R}^n)$, u bounded.

We define the fractional Laplacian

$$(-\Delta)^s u(x) := C_{n,s} \cdot \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy .$$

The fractional Laplacian can also be defined using Fourier transforms

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}\Big(\big(2\pi|\xi|\big)^{2s} \cdot \mathcal{F}(u)(\xi)\Big)(x)$$

$\boxed{\text{Theorem}}$ (Caffarelli-Silvestre)

Let $D = \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,+\infty) \ni (x,z)$; we identify $\partial \mathbb{R}^{n+1}_+ \cong \mathbb{R}^n$

Let $u \in C^2(\mathbb{R}^n)$, u bounded. Let $U : \overline{D} \to \mathbb{R}$ be the solution of

$$(\text{D.P.})\begin{cases} \operatorname{div}\left(z^{1-2s}\nabla U\right) = 0 & \text{in } D=\mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

which is equivalent to

$$(\text{D.P.}')\begin{cases} \left(\frac{1-2s}{2z} + \frac{1}{2}\Delta_{x,z}\right) U = 0 & \text{in } D=\mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Then

$$(-\Delta)^s u(x) = -A_{n,s} \cdot \lim_{z \to 0} z^{1-2s}\, \partial_z U(x,z) \qquad (\text{E}9)$$

Caffarelli and Silvestre proved that this theorem follows from the following formula about a Poisson-type Kernel

[Caffarelli, Silvestre. Section 3]

$\boxed{\text{Proposition}}$ (Poisson Kernel formula)

Let $u \in C^2(\mathbb{R}^n)$, u bounded. Let $D = \mathbb{R}^{n+1}_+$, $\mathbb{R}^n \triangleq \partial D$.
Let U be a solution of

$$
(D.P.')\begin{cases} \left(\dfrac{1-2s}{2z}\dfrac{\partial}{\partial z} + \dfrac{1}{2}\Delta_{x,z}\right)U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{for } x \in \mathbb{R}^n \end{cases}
$$

Then

$$U(x_0, z_0) = (K_{z_0} * u)(x_0) = \int_{\mathbb{R}^n} K_{z_0}(y - x_0) \cdot u(y)\, dy \quad (E10)$$

where

$$K_z(x) = C_{n,s} \cdot \frac{z^{2s}}{(|x|^2 + z^2)^{\frac{n}{2}+s}} \quad (E11)$$

We are going to prove this proposition using stochastic analysis.

$\boxed{\text{Proof}}$ Consider the Itô diffusion in $\mathbb{R}^{n+1}$ $Y = \{Y_t\}_{t \geq 0}$, satisfying

$$
\begin{pmatrix} dY_t^1 \\ dY_t^2 \\ \vdots \\ dY_t^n \\ dY_t^{n+1} \end{pmatrix} =: \begin{pmatrix} dX_t^1 \\ dX_t^2 \\ \vdots \\ dX_t^n \\ dZ_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1-2s}{2Z_t} \end{pmatrix} dt + I_{n+1} \cdot \begin{pmatrix} dB_t^1 \\ dB_t^2 \\ \vdots \\ dB_t^n \\ dB_t^{n+1} \end{pmatrix}
$$

where $(B_t^1, \dots, B_t^{m+1})$ is a standard $(m+1)$-dimensional Brownian motion. Let $y_0 = (x_0, z_0)$ be the starting point of Y.

Then Y can be written as $Y = (X, Z)$ where:

$X = \{X_t\}_{t \geq 0}$ is a standard m-dimensional Brownian motion starting from x_0,

$Z = \{Z_t\}_{t \geq 0}$ is a Bessel process starting from z_0 and independent from X.

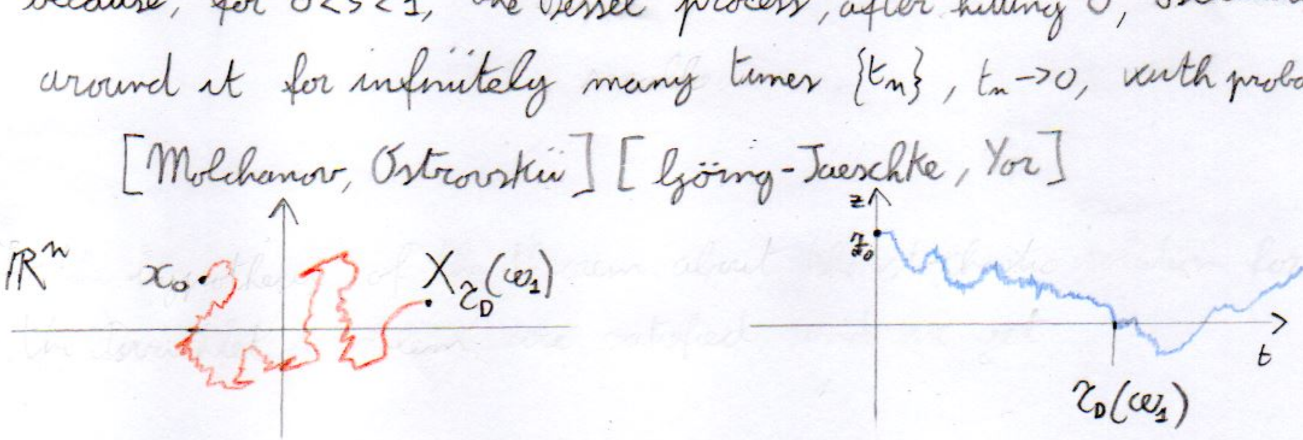
Using the characterization of the infinitesimal generator of diffusions we get that the infinitesimal generator of Y is

$$\mathcal{A} f(x, z) = \left(\frac{1-2s}{2z} \frac{\partial}{\partial z} + \frac{1}{2} \Delta_{x,z} \right) f(x, z)$$

$\mathcal{A}$ is the operator associated to (D.P.').

Moreover, $y = (x, 0) \in \partial \mathbb{R}_+^{m+1}$ is always regular w.r.t. $\{Y_t\}_{t \geq 0}$ because, for $0 < s < 1$, the Bessel process, after hitting 0, oscillates around it for infinitely many times $\{t_n\}$, $t_n \to 0$, with probability 1.

[Molchanov, Ostrovskii] [Göing-Jaeschke, Yor]



On the left: the trace of a trajectory of X up to the exit time τ_D

On the right: the graph of a trajectory of Z over the time axis.

So we proved that the hypotheses of the theorem about the stochastic solution to (D.P.') are satisfied, so we have that

$$w(x_0, z_0) := \mathbb{E}^{(x_0, z_0)}\left[u\left(Y_{\tau_0}\right)\right] \quad \text{for } (x_0, z_0) \in \overline{\mathbb{R}^{n+1}_+}$$

satisfies

$$(D.P.')\begin{cases}\left(\frac{1-2s}{2z}\frac{\partial}{\partial z} + \frac{1}{2}\Delta_{x,z}\right)w = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u(x,0) = u(x) & \text{for } x \in \mathbb{R}^n\end{cases}$$

However, the solution of this problem is unique [Øksendal, Theorem 9.3.3], so we have

$$U(x_0, z_0) = \mathbb{E}^{(x_0, z_0)}\left[u\left(Y_{\tau_0}\right)\right].$$

Now we compute the expected value for a starting point (x_0, z_0), $z_0>0$.

Let $(\Omega_1, \mathcal{F}_1, P_1)$ be the probability space domain of $X_t : \Omega_1 \to \mathbb{R}^n$

Let $(\Omega_2, \mathcal{F}_2, P_2)$ be the probability space domain of $Z_t : \Omega_2 \to \mathbb{R}$

Then the domain of the random variables $Y_t = (X_t, Z_t)$ is the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \times P_2)$. We compute

$$U(x_0, z_0) = \int_{\Omega_1 \times \Omega_2} u\left(Y^{(x_0, z_0)}_{\tau_0(\omega_1, \omega_2)}(\omega_1, \omega_2)\right) d(P_1 \times P_2)(\omega_1, \omega_2)$$

We are going to write, with a little abuse of notation,

$$u\left(Y^{(x_0, z_0)}_{\tau_D(\omega_1, \omega_2)}(\omega_1, \omega_2)\right) = u\left(X^{x_0}_{\tau_D(\omega_2)}(\omega_1)\right)$$

because the last component of Y_{τ_D} is equal to 0 almost surely, and the first n components of Y represent a Brownian motion independent from ω_2, and τ_D depends only on the last component of Y: the Bessel process Z.

We apply Fubini-Tonelli theorem and we get

$$U(x_0, z_0) = \int_{\Omega_2}\left[\int_{\Omega_1} u\left(X^{x_0}_{\tau_D(\omega_2)}(\omega_1)\right) dP_1(\omega_1)\right] dP_2(\omega_2)$$

However, $X^{x_0}_{\tau_D(\omega_2)} \sim N_{x_0, \tau_D(\omega_2)\cdot I_n}$, so we use the density of the multinormal variable to get

$$U(x_0, z_0) = \int_{\Omega_2}\left[\int_{\Omega_1} \frac{1}{(2\pi\, \tau_D(\omega_2))^{\frac{n}{2}}} \cdot e^{-\frac{|y-x_0|^2}{2\tau_D(\omega_2)}}\, u(y)\, dy\right] dP_2(\omega_2)$$

Now we use the density of the variable τ_D from the equation (E7) to get

$$U(x_0,z_0)=\int_0^{+\infty}\frac{1}{t\Gamma(s)}\left(\frac{z_0^2}{2t}\right)^s e^{-\frac{z_0^2}{2t}}\cdot\int_{\mathbb{R}^n}\frac{1}{(2\pi t)^{\frac{n}{2}}}e^{-\frac{|y-x_0|^2}{2t}}u(y)\,dy\,dt$$

We change the order of integration and rearrange the factors to get

$$U(x_0,z_0)=\int_{\mathbb{R}^n}\left(\frac{1}{\pi}\right)^{\frac{n}{2}}\frac{\Gamma(s+\frac{n}{2})}{\Gamma(s)}\cdot\frac{z_0^{2s}}{(|y-x_0|^2+z_0^2)^{s+\frac{n}{2}}}u(y)\cdot$$

$$\cdot\left[\int_0^{+\infty}\frac{1}{t\Gamma(s+\frac{n}{2})}\left(\frac{|y-x_0|^2+z_0^2}{2t}\right)^{s+\frac{n}{2}}e^{-\frac{|y-x_0|^2+z_0^2}{2t}}dt\right]dy$$

However, we have shown that $\forall\alpha>0\ \forall M>0$

$$\int_0^{+\infty}\frac{1}{t\cdot\Gamma(\alpha)}\left(\frac{M}{2t}\right)^{\alpha}e^{-\frac{M}{2t}}dt=1,$$ so the integral in dt is equal to 1. $\begin{pmatrix} M=|y-x_0|^2+z_0^2 \\ \alpha=s+\frac{n}{2}\end{pmatrix}$

So we simplify and get

$$U(x_0,z_0)=\int_{\mathbb{R}^n}\left(\frac{1}{\pi}\right)^{\frac{n}{2}}\frac{\Gamma(s+\frac{n}{2})}{\Gamma(s)}\frac{z_0^{2s}}{(|y-x_0|^2+z_0^2)^{s+\frac{n}{2}}}u(y)\,dy.$$

We define $K_z(x):=\left(\frac{1}{\pi}\right)^{\frac{n}{2}}\frac{\Gamma(s+\frac{n}{2})}{\Gamma(s)}\frac{z^{2s}}{(|x|^2+z^2)^{s+\frac{n}{2}}}$ and we get

$$U(x_0,z_0)=\int_{\mathbb{R}^n}K_{z_0}(y-x_0)u(y)\,dy=(K_{z_0}*u)(x_0).$$

So $C_{n,s}=\left(\frac{1}{\pi}\right)^{\frac{n}{2}}\frac{\Gamma(s+\frac{n}{2})}{\Gamma(s)}$, and the proof is finished.

$\square$

Thank you for your attention.

References:

- B. Øksendal. Stochastic differential equations
- S. A. Molchanov, E. Ostrovskii. Symmetric stable processes as traces of degenerate diffusion processes
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